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Vertex operator algebra analogue of embedding of B_4 into F_4

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Abstract

Let $L_B(-\frac{5}{2}, 0)$ (resp. $L_F(-\frac{5}{2}, 0)$) be the simple vertex operator algebra associated with affine Lie algebra of type $B_4^{(1)}$ (resp. $F_4^{(1)}$) with the lowest admissible half-integer level $-\frac{5}{2}$. We show that $L_B(-\frac{5}{2}, 0)$ is a vertex subalgebra of $L_F(-\frac{5}{2}, 0)$ with the same conformal vector, and that $L_F(-\frac{5}{2}, 0)$ is isomorphic to the extension of $L_B(-\frac{5}{2}, 0)$ by its only irreducible module other than itself. We also study the representation theory of $L_F(-\frac{5}{2}, 0)$, and determine the decompositions of irreducible weak $L_F(-\frac{5}{2}, 0)$-modules from the category $\mathcal{O}$ into direct sums of irreducible weak $L_B(-\frac{5}{2}, 0)$-modules.

MSC: 17B69; 17B67

1. Introduction

The embedding of B_4 into F_4 has been studied on the level of Lie groups and Lie algebras both in mathematics and physics (cf. [6,17,24]). It is particularly interesting that simple Lie algebra of type F_4 contains three copies of simple Lie algebra of type B_4 as Lie subalgebras.

Let $\mathfrak{g}_F$ be the simple Lie algebra of type F_4, and $\mathfrak{g}_B$ its Lie subalgebra of type B_4. The decomposition of $\mathfrak{g}_F$ into a direct sum of irreducible $\mathfrak{g}_B$-modules is:

$$\mathfrak{g}_F \cong \mathfrak{g}_B \oplus V_B(\bar{\omega}_4), \tag{1}$$

where $V_B(\bar{\omega}_4)$ is the irreducible highest weight $\mathfrak{g}_B$-module whose highest weight is the fundamental weight $\bar{\omega}_4$ for $\mathfrak{g}_B$.

We want to study the analogue of relation (1) in the case of vertex operator algebras associated with affine Lie algebras. Let $\hat{\mathfrak{g}}_B$ be the affine Lie algebra of type $B_4^{(1)}$, and $\hat{\mathfrak{g}}_F$ the affine Lie algebra of type $F_4^{(1)}$. For $k \in \mathbb{C}$, denote by $L_B(k, 0)$ and $L_F(k, 0)$ simple vertex operator algebras associated with $\hat{\mathfrak{g}}_B$ and $\hat{\mathfrak{g}}_F$ of level k, respectively, with conformal vectors obtained from Segal–Sugawara construction.

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We want to study a certain level k, such that $L_B(k,0)$ is a vertex subalgebra of $L_F(k,0)$ with the same conformal vector. The equality of conformal vectors implies the equality of corresponding central charges

$$\frac{k\dim\mathfrak{g}_B}{k+h_B^\vee}=\frac{k\dim\mathfrak{g}_F}{k+h_F^\vee}.$$

This is a linear equation in k, and its only solution is $k=-\frac{5}{2}$. This suggests the study of vertex operator algebras $L_B(-\frac{5}{2},0)$ and $L_F(-\frac{5}{2},0)$.

Vertex operator algebra $L_B(-\frac{5}{2},0)$ is a special case of vertex operator algebras associated with type B affine Lie algebras of admissible half-integer levels, studied in [25]. Admissible weights are a class of weights defined in [19, 20], that includes dominant integral weights. The character formula for admissible modules is given in these papers, which inspires the study of vertex operator algebras with admissible levels. In case of non-integer admissible levels, these vertex operator algebras are not rational in the sense of Zhu [28], but in all known cases, they have properties similar to rationality in the category of weak modules that are in category $\mathcal{O}$ as modules for corresponding affine Lie algebra (cf. [1,4,11]). Admissible modules for affine Lie algebras were also recently studied in [3,12,16,27].

In case of affine Lie algebra $\hat{\mathfrak{g}}_B$ of type $B_4^{(1)}$, levels $n-\frac{7}{2}$ are studied in [25], for $n\in\mathbb{N}$. For certain subcategories of the category of weak $L_B(n-\frac{7}{2},0)$-modules, the irreducible objects are classified and semisimplicity is proved. It follows that in case $n=1$, only irreducible $L_B(-\frac{5}{2},0)$-modules are $L_B(-\frac{5}{2},0)$ and $L_B(-\frac{5}{2},\bar{\omega}_4)$.

Another motivation for studying this case is a problem of extension of a vertex operator algebra by a module. This problem was studied in [10], where all simple current extensions of vertex operator algebras associated with affine Lie algebras (except for $E_8^{(1)}$) of positive integer levels k are constructed. These extensions have a structure of abelian intertwining algebra, defined and studied in [9]. In some special cases, these extensions have a structure of a vertex operator algebra. In our case of admissible non-integer level $k=-\frac{5}{2}$, it is natural to consider the extension of vertex operator algebra $L_B(-\frac{5}{2},0)$ by its only irreducible module other then itself, $L_B(-\frac{5}{2},\bar{\omega}_4)$.

In this paper we study vertex operator algebras $L_F(n-\frac{7}{2},0)$, for $n\in\mathbb{N}$. We show that levels $n-\frac{7}{2}$ are admissible for $\hat{\mathfrak{g}}_F$. Results on admissible modules from [19] imply that $L_F(n-\frac{7}{2},0)$ is a quotient of the generalized Verma module by the maximal ideal generated by one singular vector. We determine the formula for that singular vector and using that, we show that $L_F(n-\frac{7}{2},0)$ contains three copies of $L_B(n-\frac{7}{2},0)$ as vertex subalgebras.

In the case $n=1$, we show that these three copies of $L_B(-\frac{5}{2},0)$ have the same conformal vector as $L_F(-\frac{5}{2},0)$, which along with results from [25] implies that

$$L_F\left(-\frac{5}{2},0\right)\cong L_B\left(-\frac{5}{2},0\right)\oplus L_B\left(-\frac{5}{2},\bar{\omega}_4\right),$$

which is a vertex operator algebra analogue of relation (1). This also implies that the extension of vertex operator algebra $L_B(-\frac{5}{2},0)$ by its module $L_B(-\frac{5}{2},\bar{\omega}_4)$ has a structure of vertex operator algebra, isomorphic to $L_F(-\frac{5}{2},0)$.

Furthermore, we study the category of weak $L_F(-\frac{5}{2},0)$-modules that are in category $\mathcal{O}$ as $\hat{\mathfrak{g}}_F$-modules. Using three copies of $L_B(-\frac{5}{2},0)$ contained in $L_F(-\frac{5}{2},0)$, methods from [23,2,4] and results from [25], we obtain the classification of irreducible objects in that category. Using results from [20] we prove semisimplicity of that category. We also determine decompositions of irreducible weak $L_F(-\frac{5}{2},0)$-modules from category $\mathcal{O}$ into direct sums of irreducible weak $L_B(-\frac{5}{2},0)$-modules. It turns out that these direct sums are finite.

We also study the category of $L_F(n-\frac{7}{2},0)$-modules with higher admissible half-integer levels $n-\frac{7}{2}$, for $n\in\mathbb{N}$. Using results from [25], we classify irreducible objects in that category and prove semisimplicity of that category.

2. Vertex operator algebras associated with affine Lie algebras

This section is preliminary. We recall some necessary definitions and fix the notation. We review certain results about vertex operator algebras and corresponding modules. The emphasis is on the class of vertex operator algebras associated with affine Lie algebras, because we study special cases in that class.

2.1. Vertex operator algebras and modules

Let $(V, Y, \mathbf{1}, \omega)$ be a vertex operator algebra (cf. [8,13,14,21]). Specially, the triple $(V, Y, \mathbf{1})$ carries the structure of a vertex algebra.

A *vertex subalgebra* of vertex algebra V is a subspace U of V such that $\mathbf{1} \in U$ and $Y(a, z)U \subseteq U[[z, z^{-1}]]$ for any $a \in U$. Suppose that $(V, Y, \mathbf{1}, \omega)$ is a vertex operator algebra and $(U, Y, \mathbf{1}, \omega')$ a vertex subalgebra of V, that has a structure of vertex operator algebra. We say that U is a *vertex operator subalgebra* of V if $\omega' = \omega$.

An *ideal* in a vertex operator algebra V is a subspace I of V satisfying $Y(a, z)I \subseteq I[[z, z^{-1}]]$ for any $a \in V$. Given an ideal I in V, such that $\mathbf{1} \notin I$, $\omega \notin I$, the quotient V/I admits a natural vertex operator algebra structure.

Let (M, Y_M) be a weak module for a vertex operator algebra V (cf. [22]). A $\mathbb{Z}_+$-*graded weak* V-module [15] is a weak V-module M together with a $\mathbb{Z}_+$-gradation $M = \oplus_{n=0}^{\infty} M(n)$ such that

$$a_m M(n) \subseteq M(n + r - m - 1) \quad \text{for } a \in V_{(r)}, m, n, r \in \mathbb{Z},$$

where $M(n) = 0$ for $n < 0$ by definition.

A weak V-module M is called a V-*module* if $L(0)$ acts semisimply on M with the decomposition into $L(0)$-eigenspaces $M = \oplus_{\alpha \in \mathbb{C}} M_{(\alpha)}$ such that for any $\alpha \in \mathbb{C}$, $\dim M_{(\alpha)} < \infty$ and $M_{(\alpha+n)} = 0$ for $n \in \mathbb{Z}$ sufficiently small.

2.2. Zhu's $A(V)$ theory

Let V be a vertex operator algebra. Following [28], we define bilinear maps $* : V \times V \to V$ and $\circ : V \times V \to V$ as follows. For any homogeneous $a \in V$ and for any $b \in V$, let

$$a \circ b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt} a}}{z^2} Y(a, z)b$$
$$a * b = \mathrm{Res}_z \frac{(1+z)^{\mathrm{wt} a}}{z} Y(a, z)b$$

and extend to $V \times V \to V$ by linearity. Denote by $O(V)$ the linear span of elements of the form $a \circ b$, and by $A(V)$ the quotient space $V/O(V)$. For $a \in V$, denote by $[a]$ the image of a under the projection of V onto $A(V)$. The multiplication $*$ induces the multiplication on $A(V)$ and $A(V)$ has a structure of an associative algebra.

Proposition 1 (*[15, Proposition 1.4.2]*). *Let I be an ideal of V. Assume $\mathbf{1} \notin I$, $\omega \notin I$. Then the associative algebra $A(V/I)$ is isomorphic to $A(V)/A(I)$, where $A(I)$ is the image of I in $A(V)$.*

For any homogeneous $a \in V$ we define $o(a) = a_{\mathrm{wt} a - 1}$ and extend this map linearly to V.

Proposition 2 (*[28, Theorems 2.1.2 and 2.2.1]*).

(a) *Let $M = \oplus_{n=0}^{\infty} M(n)$ be a $\mathbb{Z}_+$-graded weak V-module. Then $M(0)$ is an $A(V)$-module defined as follows:*

$$[a].v = o(a)v,$$

for any $a \in V$ and $v \in M(0)$.

(b) *Let U be an $A(V)$-module. Then there exists a $\mathbb{Z}_+$-graded weak V-module M such that the $A(V)$-modules $M(0)$ and U are isomorphic.*

Proposition 3 (*[28, Theorem 2.2.2]*). *The equivalence classes of the irreducible $A(V)$-modules and the equivalence classes of the irreducible $\mathbb{Z}_+$-graded weak V-modules are in one-to-one correspondence.*

2.3. Modules for affine Lie algebras

Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with a triangular decomposition $\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$. Let Δ be the root system of $(\mathfrak{g}, \mathfrak{h})$, $\Delta_+ \subset \Delta$ the set of positive roots, θ the highest root and $(\cdot, \cdot) : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ the Killing form, normalized by the condition $(\theta, \theta) = 2$.

The affine Lie algebra $\hat{\mathfrak{g}}$ associated with $\mathfrak{g}$ is the vector space $\mathfrak{g} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c$ equipped with the usual bracket operation and canonical central element c (cf. [18]). Let $h^\vee$ be the dual Coxeter number of $\hat{\mathfrak{g}}$. Let $\hat{\mathfrak{g}} = \hat{\mathfrak{n}}_- \oplus \hat{\mathfrak{h}} \oplus \hat{\mathfrak{n}}_+$ be the corresponding triangular decomposition of $\hat{\mathfrak{g}}$. Denote by $\hat{\Delta}$ the set of roots of $\hat{\mathfrak{g}}$, by $\hat{\Delta}^{\mathrm{re}}$ the set of real roots of $\hat{\mathfrak{g}}$, and by $\alpha^\vee$ denote the co-root of a real root $\alpha \in \hat{\Delta}^{\mathrm{re}}$.

For every weight $\lambda \in \hat{\mathfrak{h}}^*$, denote by $M(\lambda)$ the Verma module for $\hat{\mathfrak{g}}$ with highest weight λ, and by $L(\lambda)$ the irreducible $\hat{\mathfrak{g}}$-module with highest weight λ.

Let U be a $\mathfrak{g}$-module, and let $k \in \mathbb{C}$. Let $\hat{\mathfrak{g}}_+ = \mathfrak{g} \otimes t\mathbb{C}[t]$ act trivially on U and c as scalar k. Considering U as a $\mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+$-module, we have the induced $\hat{\mathfrak{g}}$-module (so called *generalized Verma module*)

$$N(k, U) = U(\hat{\mathfrak{g}}) \otimes_{U(\mathfrak{g} \oplus \mathbb{C}c \oplus \hat{\mathfrak{g}}_+)} U.$$

For a fixed $\mu \in \mathfrak{h}^*$, denote by $V(\mu)$ the irreducible highest weight $\mathfrak{g}$-module with highest weight μ.

We shall use the notation $N(k, \mu)$ to denote the $\hat{\mathfrak{g}}$-module $N(k, V(\mu))$. Denote by $J(k, \mu)$ the maximal proper submodule of $N(k, \mu)$ and by $L(k, \mu)$ the corresponding irreducible quotient $N(k, \mu)/J(k, \mu)$.

2.4. Admissible modules for affine Lie algebras

Let $\hat{\Delta}^{\vee\mathrm{re}}$ (resp. $\hat{\Delta}_+^{\vee\mathrm{re}}$) $\subset \hat{\mathfrak{h}}$ be the set of real (resp. positive real) co-roots of $\hat{\mathfrak{g}}$. Fix $\lambda \in \hat{\mathfrak{h}}^*$. Let $\hat{\Delta}_\lambda^{\vee\mathrm{re}} = \{\alpha \in \hat{\Delta}^{\vee\mathrm{re}} \mid \langle \lambda, \alpha \rangle \in \mathbb{Z}\}$, $\hat{\Delta}_{\lambda+}^{\vee\mathrm{re}} = \hat{\Delta}_\lambda^{\vee\mathrm{re}} \cap \hat{\Delta}_+^{\vee\mathrm{re}}$, $\hat{\Pi}^\vee$ the set of simple co-roots in $\hat{\Delta}^{\vee\mathrm{re}}$ and $\hat{\Pi}_\lambda^\vee = \{\alpha \in \hat{\Delta}_{\lambda+}^{\vee\mathrm{re}} \mid \alpha$ not equal to a sum of several co-roots from $\hat{\Delta}_{\lambda+}^{\vee\mathrm{re}}\}$. Define ρ in the usual way, and denote by $w.\lambda$ the "shifted" action of an element w of the Weyl group of $\hat{\mathfrak{g}}$.

Recall that a weight $\lambda \in \hat{\mathfrak{h}}^*$ is called *admissible* (cf. [19,20,27]) if the following properties are satisfied:

$$\langle \lambda + \rho, \alpha \rangle \notin -\mathbb{Z}_+ \quad \text{for all } \alpha \in \hat{\Delta}_+^{\vee\mathrm{re}},$$
$$\mathbb{Q}\hat{\Delta}_\lambda^{\vee\mathrm{re}} = \mathbb{Q}\hat{\Pi}^\vee.$$

The irreducible $\hat{\mathfrak{g}}$-module $L(\lambda)$ is called admissible if the weight $\lambda \in \hat{\mathfrak{h}}^*$ is admissible.

We shall use the following results of V. Kac and M. Wakimoto:

Proposition 4 (*[19, Corollary 2.1]*). *Let λ be an admissible weight. Then*

$$L(\lambda) \cong \frac{M(\lambda)}{\sum_{\alpha \in \hat{\Pi}_\lambda^\vee} U(\hat{\mathfrak{g}}) v^\alpha},$$

where $v^\alpha \in M(\lambda)$ is a singular vector of weight $r_\alpha.\lambda$, the highest weight vector of $M(r_\alpha.\lambda) = U(\hat{\mathfrak{g}})v^\alpha \subset M(\lambda)$.

Proposition 5 (*[20, Theorem 4.1]*). *Let M be a $\hat{\mathfrak{g}}$-module from the category $\mathcal{O}$ such that for any irreducible subquotient $L(\nu)$ the weight ν is admissible. Then M is completely reducible.*

2.5. Vertex operator algebras $N(k, 0)$ and $L(k, 0)$, for $k \neq -h^\vee$

Since $V(0)$ is the one-dimensional trivial $\mathfrak{g}$-module, it can be identified with $\mathbb{C}$. Denote by $\mathbf{1} = 1 \otimes 1 \in N(k, 0)$. We note that $N(k, 0)$ is spanned by the elements of the form $x_1(-n_1 - 1) \cdots x_m(-n_m - 1)\mathbf{1}$, where $x_1, \ldots, x_m \in \mathfrak{g}$ and $n_1, \ldots, n_m \in \mathbb{Z}_+$, with $x(n)$ denoting the representation image of $x \otimes t^n$ for $x \in \mathfrak{g}$ and $n \in \mathbb{Z}$. Vertex operator map $Y(\cdot, z) : N(k, 0) \to (\mathrm{End}\, N(k, 0))[[z, z^{-1}]]$ is uniquely determined by defining $Y(\mathbf{1}, z)$ to be the identity operator on $N(k, 0)$ and

$$Y(x(-1)\mathbf{1}, z) = \sum_{n \in \mathbb{Z}} x(n) z^{-n-1},$$

for $x \in \mathfrak{g}$. In the case that $k \neq -h^\vee$, $N(k, 0)$ has a conformal vector

$$\omega = \frac{1}{2(k + h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} a^i(-1) b^i(-1)\mathbf{1}, \tag{2}$$

where $\{a^i\}_{i=1,\dots,\dim \mathfrak{g}}$ is an arbitrary basis of $\mathfrak{g}$, and $\{b^i\}_{i=1,\dots,\dim \mathfrak{g}}$ the corresponding dual basis of $\mathfrak{g}$ with respect to the form $(\cdot,\cdot)$. We have the following result from [15,22]:

Proposition 6. *If $k \neq -h^\vee$, the quadruple $(N(k,0), Y, \mathbf{1}, \omega)$ defined above is a vertex operator algebra.*

The associative algebra $A(N(k,0))$ is identified in the next proposition:

Proposition 7 (*[15, Theorem 3.1.1]*). *The associative algebra $A(N(k,0))$ is canonically isomorphic to $U(\mathfrak{g})$. The isomorphism is given by $F : A(N(k,0)) \to U(\mathfrak{g})$*

$$F([x_1(-n_1-1)\cdots x_m(-n_m-1)\mathbf{1}]) = (-1)^{n_1+\cdots+n_m} x_m \cdots x_1,$$

for any $x_1,\dots,x_m \in \mathfrak{g}$ and any $n_1,\dots,n_m \in \mathbb{Z}_+$.

For any $\mu \in \mathfrak{h}^*$, $N(k,\mu)$ is a $\mathbb{Z}_+$-graded weak $N(k,0)$-module, and $L(k,\mu)$ is an irreducible $\mathbb{Z}_+$-graded weak $N(k,0)$-module. Denote by $v_{k,\mu}$ the highest weight vector of $L(k,\mu)$. Then the lowest conformal weight of $L(k,\mu)$ is given by relation

$$L(0)v_{k,\mu} = \frac{(\mu,\mu+2\bar{\rho})}{2(k+h^\vee)} v_{k,\mu}, \tag{3}$$

where $\bar{\rho}$ is the sum of fundamental weights of $\mathfrak{g}$.

Since every $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is also an ideal in the vertex operator algebra $N(k,0)$, it follows that $L(k,0)$ is a vertex operator algebra, for any $k \neq -h^\vee$. The associative algebra $A(L(k,0))$ is identified in the next proposition, in the case when maximal $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is generated by one singular vector.

Proposition 8. *Assume that the maximal $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is generated by a singular vector, i.e. $J(k,0) = U(\hat{\mathfrak{g}})v$. Then*

$$A(L(k,0)) \cong \frac{U(\mathfrak{g})}{I},$$

where I is the two-sided ideal of $U(\mathfrak{g})$ generated by $u = F([v])$.
Let U be a $\mathfrak{g}$-module. Then U is an $A(L(k,0))$-module if and only if $IU = 0$.

2.6. Modules for associative algebra $A(L(k,0))$

In this subsection we present the method from [23,2,4] for classification of irreducible $A(L(k,0))$-modules from the category $\mathcal{O}$ by solving certain systems of polynomial equations. We assume that the maximal $\hat{\mathfrak{g}}$-submodule of $N(k,0)$ is generated by a singular vector v.

Denote by $_L$ the adjoint action of $U(\mathfrak{g})$ on $U(\mathfrak{g})$ defined by $X_L f = [X,f]$ for $X \in \mathfrak{g}$ and $f \in U(\mathfrak{g})$. Let R be a $U(\mathfrak{g})$-submodule of $U(\mathfrak{g})$ generated by the vector $u = F([v])$ under the adjoint action. Clearly, R is an irreducible highest weight $U(\mathfrak{g})$-module with the highest weight vector u. Let R_0 be the zero-weight subspace of R. The next proposition follows from [2, Proposition 2.4.1], [4, Lemma 3.4.3]:

Proposition 9. *Let $V(\mu)$ be an irreducible highest weight $U(\mathfrak{g})$-module with the highest weight vector v_μ, for $\mu \in \mathfrak{h}^*$. The following statements are equivalent:*

(1) *$V(\mu)$ is an $A(L(k,0))$-module,*
(2) *$RV(\mu) = 0$,*
(3) *$R_0 v_\mu = 0$.*

Let $r \in R_0$. Clearly there exists the unique polynomial $p_r \in S(\mathfrak{h})$ such that

$$r v_\mu = p_r(\mu) v_\mu.$$

Set $\mathcal{P}_0 = \{\, p_r \mid r \in R_0\}$. We have:

Corollary 10. *There is one-to-one correspondence between*

(1) *irreducible $A(L(k,0))$-modules from the category $\mathcal{O}$,*
(2) *weights $\mu \in \mathfrak{h}^*$ such that $p(\mu) = 0$ for all $p \in \mathcal{P}_0$.*

3. Simple Lie algebras of type B_4 and F_4

Let

$$\Delta_F = \{\pm\epsilon_i \mid 1 \le i \le 4\} \cup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 4\} \cup \left\{\frac{1}{2}(\pm\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\right\}$$

be the root system of type F_4. Fix the set of positive roots $\Delta_F^+ = \{\epsilon_i, \mid 1 \le i \le 4\} \cup \{\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 4\} \cup \{\frac{1}{2}(\epsilon_1 \pm \epsilon_2 \pm \epsilon_3 \pm \epsilon_4)\}$. Then the simple roots are

$$\alpha_1 = \epsilon_2 - \epsilon_3, \qquad \alpha_2 = \epsilon_3 - \epsilon_4, \qquad \alpha_3 = \epsilon_4, \qquad \alpha_4 = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4).$$

The highest root is $\theta = \epsilon_1 + \epsilon_2$.

The subset $\Delta_B \subseteq \Delta_F$, defined by

$$\Delta_B = \{\pm\epsilon_i \mid 1 \le i \le 4\} \cup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 4\}$$

is a root system of type B_4. Clearly, $\theta \in \Delta_B$. The corresponding simple roots are

$$\beta_1 = \epsilon_1 - \epsilon_2, \qquad \beta_2 = \epsilon_2 - \epsilon_3, \qquad \beta_3 = \epsilon_3 - \epsilon_4, \qquad \beta_4 = \epsilon_4.$$

Furthermore, subsets $\Delta_B', \Delta_B'' \subseteq \Delta_F$ defined by

$$\begin{aligned}
\Delta_B' &= \left\{\pm\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4), \pm\frac{1}{2}(\epsilon_1 + \epsilon_2 - \epsilon_3 + \epsilon_4), \pm\frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 + \epsilon_4),\right.\\
&\quad \left.\pm\,\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 - \epsilon_4)\right\} \cup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 4\},\\
\Delta_B'' &= \left\{\pm\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4), \pm\frac{1}{2}(\epsilon_1 - \epsilon_2 + \epsilon_3 - \epsilon_4), \pm\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4),\right.\\
&\quad \left.\pm\,\frac{1}{2}(\epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_4)\right\} \cup \{\pm\epsilon_i \pm \epsilon_j \mid 1 \le i < j \le 4\}
\end{aligned}$$

are also root systems of type B_4. The corresponding simple roots are

$$\beta_1' = \epsilon_3 - \epsilon_4, \qquad \beta_2' = \epsilon_2 - \epsilon_3, \qquad \beta_3' = \epsilon_3 + \epsilon_4, \qquad \beta_4' = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 - \epsilon_4)$$

for Δ_B' and

$$\beta_1'' = \epsilon_3 + \epsilon_4, \qquad \beta_2'' = \epsilon_2 - \epsilon_3, \qquad \beta_3'' = \epsilon_3 - \epsilon_4, \qquad \beta_4'' = \frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)$$

for Δ_B''.

Let $\mathfrak{g}_F$ be the simple Lie algebra associated with the root system Δ_F, and $\mathfrak{g}_B$, $\mathfrak{g}_B'$ and $\mathfrak{g}_B''$ its Lie subalgebras associated with root systems Δ_B, Δ_B' and Δ_B'', respectively. The isomorphism π' of root systems Δ_B and Δ_B' defined by

$$\pi'(\beta_i) = \beta_i', \quad i = 1, 2, 3, 4$$

induces an isomorphism of Lie algebras $\pi' : \mathfrak{g}_B \to \mathfrak{g}_B'$. Similarly, the isomorphism π'' of root systems Δ_B and Δ_B'' defined by

$$\pi''(\beta_i) = \beta_i'', \quad i = 1, 2, 3, 4$$

induces an isomorphism of Lie algebras $\pi'' : \mathfrak{g}_B \to \mathfrak{g}_B''$. Thus, we have three ways of embedding simple Lie algebra of type B_4 into simple Lie algebra of type F_4.

Let e_i, f_i, h_i, $1 \le i \le 4$ be the Chevalley generators of $\mathfrak{g}_F$. Fix the root vectors for positive roots of $\mathfrak{g}_F$:

$$e_{\frac{1}{2}(\epsilon_1 - \epsilon_2 - \epsilon_3 + \epsilon_4)} = [e_3, e_4], \qquad e_{\epsilon_2 - \epsilon_4} = [e_2, e_1], \qquad e_{\epsilon_3} = [e_2, e_3],$$

$$
\begin{aligned}
&e_{\frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)}=[e_{\epsilon_3},e_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)}],\qquad e_{\epsilon_2}=[e_{\epsilon_2-\epsilon_4},e_3],\qquad e_{\epsilon_3+\epsilon_4}=\frac{1}{2}[e_{\epsilon_3},e_3],\\
&e_{\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)}=[e_{\frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)},e_1],\qquad e_{\frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3+\epsilon_4)}=[e_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)},e_3],\\
&e_{\epsilon_2+\epsilon_4}=\frac{1}{2}[e_{\epsilon_2},e_3],\qquad e_{\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)}=[e_{\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)},e_3],\\
&e_{\epsilon_1-\epsilon_2}=\frac{1}{2}[e_{\frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)},e_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)}],\qquad e_{\epsilon_2+\epsilon_3}=\frac{1}{2}[e_{\epsilon_3},e_{\epsilon_2}],\\
&e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)}=[e_{\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3+\epsilon_4)},e_2],\qquad e_{\epsilon_1-\epsilon_3}=[e_{\epsilon_1-\epsilon_2},e_1],\\
&e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)}=[e_3,e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)}],\qquad e_{\epsilon_1-\epsilon_4}=[e_{\epsilon_1-\epsilon_3},e_2],\qquad e_{\epsilon_1}=[e_{\epsilon_1-\epsilon_4},e_3],\\
&e_{\epsilon_1+\epsilon_4}=\frac{1}{2}[e_{\epsilon_1},e_3],\qquad e_{\epsilon_1+\epsilon_3}=\frac{1}{2}[e_{\epsilon_1},e_{\epsilon_3}],\qquad e_{\epsilon_1+\epsilon_2}=\frac{1}{2}[e_{\epsilon_2},e_{\epsilon_1}].
\end{aligned}
\tag{4}
$$

To define root vectors for negative roots of $\mathfrak{g}_F$, put $f_{\alpha+\beta}=-c_{\alpha,\beta}[f_\alpha,f_\beta]$ if $e_{\alpha+\beta}=c_{\alpha,\beta}[e_\alpha,e_\beta]$. Denote by $h_\alpha=\alpha^\vee=[e_\alpha,f_\alpha]$ co-roots, for any positive root $\alpha\in\Delta_F^+$. Let $\mathfrak{g}_F=\mathfrak{n}_-\oplus\mathfrak{h}\oplus\mathfrak{n}_+$ be the corresponding triangular decomposition of $\mathfrak{g}_F$.

One can easily check that $\pi'(e_\alpha)=e_{\pi'(\alpha)}$, $\pi'(f_\alpha)=f_{\pi'(\alpha)}$, and $\pi''(e_\alpha)=e_{\pi''(\alpha)}$, $\pi''(f_\alpha)=f_{\pi''(\alpha)}$ for all $\alpha\in\Delta_B^+$.

Denote by P_+^F the set of dominant integral weights for $\mathfrak{g}_F$ and by P_+^B the set of dominant integral weights for $\mathfrak{g}_B$. Denote by $\omega_1,\ldots,\omega_4\in P_+^F$ the fundamental weights of $\mathfrak{g}_F$, defined by $\omega_i(\alpha_j^\vee)=\delta_{ij}$ for all $i,j=1,\ldots,4$, and by $\bar{\omega}_1,\ldots,\bar{\omega}_4\in P_+^B$ the fundamental weights of $\mathfrak{g}_B$, defined by $\bar{\omega}_i(\beta_j^\vee)=\delta_{ij}$ for all $i,j=1,\ldots,4$.

We shall also use the notation $V_F(\mu)$ (resp. $V_B(\mu)$) for the irreducible highest weight $\mathfrak{g}_F$-module (resp. $\mathfrak{g}_B$-module) with highest weight $\mu\in\mathfrak{h}^*$.

4. Vertex operator algebras $L_F(n-\frac{7}{2},0)$ and $L_B(n-\frac{7}{2},0)$, for $n\in\mathbb{N}$

Let $\hat{\mathfrak{g}}_F$, $\hat{\mathfrak{g}}_B$, $\hat{\mathfrak{g}}'_B$ and $\hat{\mathfrak{g}}''_B$ be affine Lie algebras associated with $\mathfrak{g}_F$, $\mathfrak{g}_B$, $\mathfrak{g}'_B$ and $\mathfrak{g}''_B$, respectively. For $k\in\mathbb{C}$, denote by $N_F(k,0)$, $N_B(k,0)$, $N'_B(k,0)$ and $N''_B(k,0)$ generalized Verma modules associated with $\hat{\mathfrak{g}}_F$, $\hat{\mathfrak{g}}_B$, $\hat{\mathfrak{g}}'_B$ and $\hat{\mathfrak{g}}''_B$ of level k, and by $L_F(k,0)$, $L_B(k,0)$, $L'_B(k,0)$ and $L''_B(k,0)$ corresponding irreducible modules.

We shall also use the notation $M_F(\lambda)$ (resp. $M_B(\lambda)$) for the Verma module for $\hat{\mathfrak{g}}_F$ (resp. $\hat{\mathfrak{g}}_B$) with highest weight $\lambda\in\hat{\mathfrak{h}}^*$, and $L_F(\lambda)$ (resp. $L_B(\lambda)$) for the irreducible $\hat{\mathfrak{g}}_F$-module (resp. $\hat{\mathfrak{g}}_B$-module) with highest weight $\lambda\in\hat{\mathfrak{h}}^*$.

Since $\mathfrak{g}_B$, $\mathfrak{g}'_B$ and $\mathfrak{g}''_B$ are Lie subalgebras of $\mathfrak{g}_F$, it follows that $\hat{\mathfrak{g}}_B$, $\hat{\mathfrak{g}}'_B$ and $\hat{\mathfrak{g}}''_B$ are Lie subalgebras of $\hat{\mathfrak{g}}_F$. Using the P-B-W theorem (which gives an embedding of the universal enveloping algebra of a Lie subalgebra into the universal enveloping algebra of a Lie algebra) we obtain embeddings of generalized Verma modules $N_B(k,0)$, $N'_B(k,0)$ and $N''_B(k,0)$ into $N_F(k,0)$. Therefore, $N_B(k,0)$, $N'_B(k,0)$ and $N''_B(k,0)$ are vertex subalgebras of $N_F(k,0)$.

Furthermore, isomorphisms π' and π'' induce isomorphisms of affine Lie algebras $\pi':\hat{\mathfrak{g}}_B\to\hat{\mathfrak{g}}'_B$, $\pi'':\hat{\mathfrak{g}}_B\to\hat{\mathfrak{g}}''_B$ and isomorphisms of vertex operator algebras $\pi':N_B(k,0)\to N'_B(k,0)$, $\pi'':N_B(k,0)\to N''_B(k,0)$. Thus, $N_F(k,0)$ contains three copies of $N_B(k,0)$ as vertex subalgebras.

4.1. Vertex operator algebra $L_B(n-\frac{7}{2},0)$, for $n\in\mathbb{N}$

In this subsection we recall some facts about vertex operator algebra $L_B(n-\frac{7}{2},0)$, for $n\in\mathbb{N}$. It was proved in [25, Theorem 11] that the maximal $\hat{\mathfrak{g}}_B$-submodule of $N_B(n-\frac{7}{2},0)$ is generated by one singular vector:

Proposition 11. *The maximal $\hat{\mathfrak{g}}_B$-submodule of $N_B(n-\frac{7}{2},0)$ is $J_B(n-\frac{7}{2},0)=U(\hat{\mathfrak{g}}_B)v_B$, where*

$$v_B=\left(-\frac{1}{4}e_{\epsilon_1}(-1)^2+e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1)+e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1+\epsilon_3}(-1)+e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1)\right)^n\mathbf{1}$$

is a singular vector in $N_B(n-\frac{7}{2},0)$.

Remark 12. The choice of root vectors for $\mathfrak{g}_B$ in (4) is slightly different from the choice in [25], but formula for the vector v_B is the same in both bases.

Thus

$$L_B\left(n-\frac{7}{2},0\right)\cong\frac{N_B\left(n-\frac{7}{2},0\right)}{U(\hat{\mathfrak{g}}_B)v_B}.$$

Corollary 13. *The associative algebra $A(L_B(n-\frac{7}{2},0))$ is isomorphic to the algebra $U(\mathfrak{g}_B)/I_B$, where I_B is the two-sided ideal of $U(\mathfrak{g}_B)$ generated by*

$$u_B=\left(-\frac{1}{4}e_{\epsilon_1}^2+e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2}+e_{\epsilon_1-\epsilon_3}e_{\epsilon_1+\epsilon_3}+e_{\epsilon_1-\epsilon_4}e_{\epsilon_1+\epsilon_4}\right)^n.$$

It follows from [25, Theorems 24 and 27] that:

Proposition 14. (1) *The set*

$$\left\{L_B\left(n-\frac{7}{2},\mu\right)\mid\mu\in P_+^B,(\mu,\epsilon_1)\le n-\frac{1}{2}\right\}$$

provides the complete list of irreducible $L_B(n-\frac{7}{2},0)$-modules.

(2) *Let M be an $L_B(n-\frac{7}{2},0)$-module. Then M is completely reducible.*

In the special case $n=1$, the following was proved ([25, Theorems 31 and 33]):

Proposition 15. (1) *The set*

$$\begin{aligned}&\left\{L_B\left(-\frac{5}{2},0\right),L_B\left(-\frac{5}{2},\bar\omega_4\right),L_B\left(-\frac{5}{2},-\frac{5}{2}\bar\omega_1\right),L_B\left(-\frac{5}{2},-\frac{7}{2}\bar\omega_1+\bar\omega_4\right),L_B\left(-\frac{5}{2},-\frac{3}{2}\bar\omega_2\right),\right.\\&L_B\left(-\frac{5}{2},-\frac{5}{2}\bar\omega_2+\bar\omega_4\right),L_B\left(-\frac{5}{2},-\frac{1}{2}\bar\omega_3\right),L_B\left(-\frac{5}{2},-\frac{3}{2}\bar\omega_3+\bar\omega_4\right),L_B\left(-\frac{5}{2},\frac{1}{2}\bar\omega_1-\frac{3}{2}\bar\omega_2\right),\\&L_B\left(-\frac{5}{2},\frac{3}{2}\bar\omega_1-\frac{5}{2}\bar\omega_2+\bar\omega_4\right),L_B\left(-\frac{5}{2},-\frac{3}{2}\bar\omega_1-\frac{1}{2}\bar\omega_3\right),L_B\left(-\frac{5}{2},-\frac{1}{2}\bar\omega_1-\frac{3}{2}\bar\omega_3+\bar\omega_4\right),\\&L_B\left(-\frac{5}{2},-\frac{1}{2}\bar\omega_2-\frac{1}{2}\bar\omega_3\right),L_B\left(-\frac{5}{2},\frac{1}{2}\bar\omega_2-\frac{3}{2}\bar\omega_3+\bar\omega_4\right),L_B\left(-\frac{5}{2},-\frac{1}{2}\bar\omega_1-\frac{1}{2}\bar\omega_2-\frac{1}{2}\bar\omega_3\right),\\&\left.L_B\left(-\frac{5}{2},-\frac{3}{2}\bar\omega_1+\frac{1}{2}\bar\omega_2-\frac{3}{2}\bar\omega_3+\bar\omega_4\right)\right\}\end{aligned}$$

provides the complete list of irreducible weak $L_B(-\frac{5}{2},0)$-modules from the category $\mathcal{O}$.

(2) *Let M be a weak $L_B(-\frac{5}{2},0)$-module from the category $\mathcal{O}$. Then M is completely reducible.*

Remark 16. It follows from Propositions 14 and 15 that there are 16 irreducible weak $L_B(-\frac{5}{2},0)$-modules from the category $\mathcal{O}$, and that only irreducible $L_B(-\frac{5}{2},0)$-modules are $L_B(-\frac{5}{2},0)$ and $L_B(-\frac{5}{2},\bar\omega_4)$.

Using isomorphisms π' and π'', and Proposition 11 we obtain

Proposition 17. (1) *The maximal $\hat{\mathfrak{g}}'_B$-submodule of $N'_B(n-\frac{7}{2},0)$ is $J'_B(n-\frac{7}{2},0)=U(\hat{\mathfrak{g}}'_B)v'_B$, where*

$$\begin{aligned}v'_B=&\left(-\frac{1}{4}e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)}(-1)^2+e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_1-\epsilon_4}(-1)+e_{\epsilon_2-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1)\right.\\&\left.+\ e_{\epsilon_3-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1)\right)^n\mathbf{1}.\end{aligned}$$

(2) *The associative algebra $A(L'_B(n-\frac{7}{2},0))$ is isomorphic to the algebra $U(\mathfrak{g}'_B)/I'_B$, where I'_B is the two-sided ideal of $U(\mathfrak{g}'_B)$ generated by*

$$u'_B=\left(-\frac{1}{4}e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3-\epsilon_4)}^2+e_{\epsilon_2+\epsilon_3}e_{\epsilon_1-\epsilon_4}+e_{\epsilon_2-\epsilon_4}e_{\epsilon_1+\epsilon_3}+e_{\epsilon_3-\epsilon_4}e_{\epsilon_1+\epsilon_2}\right)^n.$$

Proof. It can easily be checked that $\pi'(v_B) = v'_B$, which implies the claim of proposition. $\square$

Proposition 18. (1) *The maximal $\hat{\mathfrak{g}}''_B$-submodule of $N''_B(n-\frac{7}{2},0)$ is $J''_B(n-\frac{7}{2},0) = U(\hat{\mathfrak{g}}''_B)v''_B$, where*

$$v''_B = \Big(-\frac{1}{4}e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)}(-1)^2 + e_{\epsilon_2+\epsilon_3}(-1)e_{\epsilon_1+\epsilon_4}(-1) + e_{\epsilon_2+\epsilon_4}(-1)e_{\epsilon_1+\epsilon_3}(-1) \\ + e_{\epsilon_3+\epsilon_4}(-1)e_{\epsilon_1+\epsilon_2}(-1)\Big)^n \mathbf{1}.$$

(2) *The associative algebra $A(L''_B(n-\frac{7}{2},0))$ is isomorphic to the algebra $U(\mathfrak{g}''_B)/I''_B$, where I''_B is the two-sided ideal of $U(\mathfrak{g}''_B)$ generated by*

$$u''_B = \Big(-\frac{1}{4}e^2_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)} + e_{\epsilon_2+\epsilon_3}e_{\epsilon_1+\epsilon_4} + e_{\epsilon_2+\epsilon_4}e_{\epsilon_1+\epsilon_3} + e_{\epsilon_3+\epsilon_4}e_{\epsilon_1+\epsilon_2}\Big)^n.$$

Proof. It can easily be checked that $\pi''(v_B) = v''_B$, which implies the claim of proposition. $\square$

We obtain

$$L'_B\left(n-\frac{7}{2},0\right) \cong \frac{N'_B\left(n-\frac{7}{2},0\right)}{U(\hat{\mathfrak{g}}'_B)v'_B} \quad \text{and}$$
$$L''_B\left(n-\frac{7}{2},0\right) \cong \frac{N''_B\left(n-\frac{7}{2},0\right)}{U(\hat{\mathfrak{g}}''_B)v''_B}.$$

4.2. *Vertex operator algebra $L_F(n-\frac{7}{2},0)$, for $n \in \mathbb{N}$*

In this subsection we show that the maximal $\hat{\mathfrak{g}}_F$-submodule of $N_F(n-\frac{7}{2},0)$, for $n \in \mathbb{N}$, is generated by one singular vector. We need two lemmas to prove that.

Denote by λ_n the weight $(n-\frac{7}{2})\Lambda_0$. Then $N_F(n-\frac{7}{2},0)$ is a quotient of the Verma module $M_F(\lambda_n)$ and $L_F(n-\frac{7}{2},0) \cong L_F(\lambda_n)$.

Lemma 19. *The weight $\lambda_n = (n-\frac{7}{2})\Lambda_0$ is admissible for $\hat{\mathfrak{g}}_F$ and*

$$\hat{\Pi}^\vee_{\lambda_n} = \{(\delta-\epsilon_1)^\vee, \alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee\}.$$

Proof. Clearly

$$\langle \lambda_n+\rho, \alpha_i^\vee\rangle = 1 \quad \text{for } 1 \le i \le 4,$$
$$\langle \lambda_n+\rho, \alpha_0^\vee\rangle = n-\frac{5}{2},$$

which implies

$$\langle \lambda_n+\rho, (\delta-\epsilon_1)^\vee\rangle = \langle \lambda_n+\rho, 2\alpha_0^\vee+2\alpha_1^\vee+2\alpha_2^\vee+\alpha_3^\vee\rangle = 2n.$$

The claim of lemma now follows easily. $\square$

Lemma 20. *Vector*

$$v_F = \Big(-\frac{1}{4}e_{\epsilon_1}(-1)^2 + e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1) + e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1+\epsilon_3}(-1) + e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1)\Big)^n \mathbf{1}$$

is a singular vector in $N_F(n-\frac{7}{2},0)$.

Proof. Clearly $v_F \in N_B(n-\frac{7}{2},0)$, and Proposition 11 implies that

$$e_i(0).v_F = 0, \quad 1 \le i \le 3$$
$$f_\theta(1).v_F = 0,$$

since $e_1(0), e_2(0), e_3(0), f_\theta(1) \in \hat{\mathfrak{g}}_B$. It remains to show that $e_4(0).v_F = 0$, which follows immediately from the fact that $e_4(0)$ commutes with all vectors $e_{\epsilon_1}(-1), e_{\epsilon_1-\epsilon_2}(-1), e_{\epsilon_1+\epsilon_2}(-1), e_{\epsilon_1-\epsilon_3}(-1), e_{\epsilon_1+\epsilon_3}(-1), e_{\epsilon_1-\epsilon_4}(-1), e_{\epsilon_1+\epsilon_4}(-1)$. □

Theorem 21. *The maximal $\hat{\mathfrak{g}}_F$-submodule of $N_F(n-\frac{7}{2},0)$ is $J_F(n-\frac{7}{2},0) = U(\hat{\mathfrak{g}}_F)v_F$, where*

$$v_F = \left(-\frac{1}{4}e_{\epsilon_1}(-1)^2 + e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1) + e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1+\epsilon_3}(-1) + e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1)\right)^n \mathbf{1}.$$

Proof. It follows from Proposition 4 and Lemma 19 that the maximal submodule of the Verma module $M_F(\lambda_n)$ is generated by five singular vectors with weights

$$r_{\delta-\epsilon_1}.\lambda_n, \qquad r_{\alpha_1}.\lambda_n, \qquad r_{\alpha_2}.\lambda_n, \qquad r_{\alpha_3}.\lambda_n, \qquad r_{\alpha_4}.\lambda_n.$$

It follows from Lemma 20 that v_F is a singular vector with weight $\lambda_n - 2n\delta + 2n\epsilon_1 = r_{\delta-\epsilon_1}.\lambda_n$. Other singular vectors have weights

$$r_{\alpha_i}.\lambda_n = \lambda_n - \langle \lambda_n + \rho, \alpha_i^\vee \rangle \alpha_i = \lambda_n - \alpha_i, \quad 1 \le i \le 4,$$

so that the images of these vectors under the projection of $M_F(\lambda_n)$ onto $N_F(n-\frac{7}{2},0)$ are 0. Therefore, the maximal submodule of $N_F(n-\frac{7}{2},0)$ is generated by the vector v_F, i.e. $J_F(n-\frac{7}{2},0) = U(\hat{\mathfrak{g}}_F)v_F$. □

It follows that

$$L_F\left(n-\frac{7}{2},0\right) \cong \frac{N_F\left(n-\frac{7}{2},0\right)}{U(\hat{\mathfrak{g}}_F)v_F}.$$

Using Theorem 21 and Proposition 8 we can identify the associative algebra $A(L_F(n-\frac{7}{2},0))$:

Proposition 22. *The associative algebra $A(L_F(n-\frac{7}{2},0))$ is isomorphic to the algebra $U(\mathfrak{g}_F)/I_F$, where I_F is the two-sided ideal of $U(\mathfrak{g}_F)$ generated by*

$$u_F = \left(-\frac{1}{4}e_{\epsilon_1}^2 + e_{\epsilon_1-\epsilon_2}e_{\epsilon_1+\epsilon_2} + e_{\epsilon_1-\epsilon_3}e_{\epsilon_1+\epsilon_3} + e_{\epsilon_1-\epsilon_4}e_{\epsilon_1+\epsilon_4}\right)^n.$$

Since $v_F = v_B$, it follows from Theorem 21 and Proposition 11 that the embedding of $N_B(n-\frac{7}{2},0)$ into $N_F(n-\frac{7}{2},0)$ induces the embedding of $L_B(n-\frac{7}{2},0)$ into $L_F(n-\frac{7}{2},0)$. We get:

Proposition 23. *$L_B(n-\frac{7}{2},0)$ is a vertex subalgebra of $L_F(n-\frac{7}{2},0)$.*

Denote by R^F (resp. R^B) the $U(\mathfrak{g}_F)$-submodule (resp. $U(\mathfrak{g}_B)$-submodule) of $U(\mathfrak{g}_F)$ (resp. $U(\mathfrak{g}_B)$) generated by the vector u_F (resp. u_B) under the adjoint action. Let R_0^F (resp. R_0^B) be the zero-weight subspace of R^F (resp. R^B). Denote by $\mathcal{P}_0^F$ and $\mathcal{P}_0^B$ the corresponding sets of polynomials, defined in Section 2.6. Since $u_B = u_F$, we have $R^B \subseteq R^F$ and

Corollary 24.

$$\mathcal{P}_0^B \subseteq \mathcal{P}_0^F.$$

Furthermore,

Proposition 25. *$L'_B(n-\frac{7}{2},0)$ and $L''_B(n-\frac{7}{2},0)$ are vertex subalgebras of $L_F(n-\frac{7}{2},0)$.*

Proof. One can easily check that

$$\frac{1}{(2n)!} f_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)}(0)^{2n}.v_F = v'_B \quad \text{and}$$
$$\frac{1}{(2n)!} f_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)}(0)^{2n}.v_F = v''_B. \tag{5}$$

Theorem 21 and Propositions 17 and 18 now imply that the embedding of $N'_B(n-\frac{7}{2},0)$ into $N_F(n-\frac{7}{2},0)$ induces the embedding of $L'_B(n-\frac{7}{2},0)$ into $L_F(n-\frac{7}{2},0)$, and that the embedding of $N''_B(n-\frac{7}{2},0)$ into $N_F(n-\frac{7}{2},0)$ induces the embedding of $L''_B(n-\frac{7}{2},0)$ into $L_F(n-\frac{7}{2},0)$. □

We conclude that $L_F(n-\frac{7}{2},0)$ contains three copies of $L_B(n-\frac{7}{2},0)$ as vertex subalgebras.

Denote by $R^{B'}$ (resp. $R^{B''}$) the $U(\mathfrak{g}'_B)$-submodule (resp. $U(\mathfrak{g}''_B)$-submodule) of $U(\mathfrak{g}'_B)$ (resp. $U(\mathfrak{g}''_B)$) generated by the vector u'_B (resp. u''_B) under the adjoint action. Let $R_0^{B'}$ (resp. $R_0^{B''}$) be the zero-weight subspace of $R^{B'}$ (resp. $R^{B''}$). Denote by $\mathcal{P}_0^{B'}$ and $\mathcal{P}_0^{B''}$ the corresponding sets of polynomials, defined in Section 2.6. It follows from relations (5) that

$$\frac{1}{(2n)!}(f^{2n}_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)})_L u_F = u'_B \quad \text{and}$$
$$\frac{1}{(2n)!}(f^{2n}_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)})_L u_F = u''_B,$$

which implies that $R^{B'} \subseteq R^F$, $R^{B''} \subseteq R^F$ and

Corollary 26.

$$\mathcal{P}_0^{B'} \subseteq \mathcal{P}_0^F \quad \textit{and} \quad \mathcal{P}_0^{B''} \subseteq \mathcal{P}_0^F.$$

5. Vertex operator algebras $L_B(-\frac{5}{2},0)$ and $L_F(-\frac{5}{2},0)$

In this section we study the case $n=1$, vertex operator algebras $L_B(-\frac{5}{2},0)$ and $L_F(-\frac{5}{2},0)$. We show that $L_B(-\frac{5}{2},0)$ is a vertex operator subalgebra of $L_F(-\frac{5}{2},0)$, i.e. that these vertex operator algebras have the same conformal vector. We use the following lemma:

Lemma 27. *Relation*

$$\begin{aligned} 7 &\sum_{\substack{\alpha\in\Delta_B^+ \\ (\alpha,\alpha)=1}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1} + f_\alpha(-1)e_\alpha(-1)\mathbf{1}) \\ &= 4 \sum_{\substack{\alpha\in\Delta_B^+ \\ (\alpha,\alpha)=2}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1} + f_\alpha(-1)e_\alpha(-1)\mathbf{1}) + \sum_{\substack{\alpha\in\Delta_B^+ \\ (\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1} \end{aligned} \tag{6}$$

holds in $L_B(-\frac{5}{2},0)$.

Proof. In the case $n=1$, Proposition 11 implies that relation

$$\left(2f_{\epsilon_1}(0)^2 + f_{\epsilon_2}(0)^2 f_{\epsilon_1-\epsilon_2}(0)^2 + f_{\epsilon_3}(0)^2 f_{\epsilon_1-\epsilon_3}(0)^2 + f_{\epsilon_4}(0)^2 f_{\epsilon_1-\epsilon_4}(0)^2\right).$$
$$\left(-\frac{1}{4}e_{\epsilon_1}(-1)^2\mathbf{1} + e_{\epsilon_1-\epsilon_2}(-1)e_{\epsilon_1+\epsilon_2}(-1)\mathbf{1} + e_{\epsilon_1-\epsilon_3}(-1)e_{\epsilon_1+\epsilon_3}(-1)\mathbf{1} + e_{\epsilon_1-\epsilon_4}(-1)e_{\epsilon_1+\epsilon_4}(-1)\mathbf{1}\right) = 0$$

holds in $L_B(-\frac{5}{2},0)$. From this relation we get the claim of lemma. □

Theorem 28. *Denote by ω_F the conformal vector for vertex operator algebra $L_F(-\frac{5}{2},0)$, and by ω_B the conformal vector for $L_B(-\frac{5}{2},0)$. Then*

$$\omega_F = \omega_B.$$

Proof. Sets $\{\frac{1}{2}h_\alpha \mid \alpha \in \Delta_B^+, (\alpha,\alpha)=1\}$, $\{\frac{1}{2}h_\alpha \mid \alpha \in \Delta_B'^+, (\alpha,\alpha)=1\}$ and $\{\frac{1}{2}h_\alpha \mid \alpha \in \Delta_B''^+, (\alpha,\alpha)=1\}$ are three orthonormal bases of $\mathfrak{h}$ with respect to the form $(\cdot,\cdot)$. Clearly,

$$\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1} = \sum_{\substack{\alpha\in\Delta_B'^+\\(\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1} = \sum_{\substack{\alpha\in\Delta_B''^+\\(\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1}. \tag{7}$$

Applying isomorphisms π' and π'' to relation (6) we obtain that relation

$$\begin{aligned}
&7\sum_{\substack{\alpha\in\Delta_B'^+\\(\alpha,\alpha)=1}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\\
&\quad= 4\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=2}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1}) + \sum_{\substack{\alpha\in\Delta_B'^+\\(\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1}
\end{aligned}\tag{8}$$

holds in the vertex subalgebra $L'_B(-\frac{5}{2},0)$ of $L_F(-\frac{5}{2},0)$, and that relation

$$\begin{aligned}
&7\sum_{\substack{\alpha\in\Delta_B''^+\\(\alpha,\alpha)=1}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\\
&\quad= 4\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=2}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1}) + \sum_{\substack{\alpha\in\Delta_B''^+\\(\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1}
\end{aligned}\tag{9}$$

holds in the vertex subalgebra $L''_B(-\frac{5}{2},0)$ of $L_F(-\frac{5}{2},0)$. Using (7) we obtain that

$$\begin{aligned}
&\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=1}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\\
&\quad= \sum_{\substack{\alpha\in\Delta_B'^+\\(\alpha,\alpha)=1}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\\
&\quad= \sum_{\substack{\alpha\in\Delta_B''^+\\(\alpha,\alpha)=1}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\\
&\quad= \frac{4}{7}\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=2}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1}) + \frac{1}{7}\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1}
\end{aligned}\tag{10}$$

holds in $L_F(-\frac{5}{2},0)$. It follows from relation (2) that

$$\begin{aligned}
\omega_F &= \frac{1}{13}\left(\frac{1}{4}\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1} + \sum_{\alpha\in\Delta_F^+}\frac{(\alpha,\alpha)}{2}(e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\right)\\
&= \frac{1}{13}\left(\frac{1}{4}\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=1}} h_\alpha(-1)^2\mathbf{1} + \sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=2}} (e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\right.
\end{aligned}$$

$$+\frac{1}{2}\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=1}}(e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})+\frac{1}{2}\sum_{\substack{\alpha\in\Delta_B'^+\\(\alpha,\alpha)=1}}(e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})$$

$$\left.+\frac{1}{2}\sum_{\substack{\alpha\in\Delta_B''^+\\(\alpha,\alpha)=1}}(e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\right).$$

Using relation (10), we obtain

$$\begin{aligned}\omega_F &= \frac{1}{28}\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=1}}h_\alpha(-1)^2\mathbf{1}+\frac{1}{7}\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=2}}(e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\\ &= \frac{1}{9}\left(\frac{1}{4}\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=1}}h_\alpha(-1)^2\mathbf{1}+\sum_{\substack{\alpha\in\Delta_B^+\\(\alpha,\alpha)=2}}(e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\right.\\ &\quad\left.+\frac{1}{2}\sum_{\substack{\alpha\in\Delta_{B+}\\(\alpha,\alpha)=1}}(e_\alpha(-1)f_\alpha(-1)\mathbf{1}+f_\alpha(-1)e_\alpha(-1)\mathbf{1})\right)=\omega_B.\quad\square\end{aligned}$$

Thus, $L_B(-\frac{5}{2},0)$ is a vertex operator subalgebra of $L_F(-\frac{5}{2},0)$.

Remark 29. Denote by ω_B' the conformal vector for vertex operator algebra $L_B'(-\frac{5}{2},0)$, and by ω_B'' the conformal vector for $L_B''(-\frac{5}{2},0)$. Relation (10) implies that

$$\omega_B'=\omega_B''=\omega_B=\omega_F$$

in $L_F(-\frac{5}{2},0)$. Thus, $L_B'(-\frac{5}{2},0)$ and $L_B''(-\frac{5}{2},0)$ are vertex operator subalgebras of $L_F(-\frac{5}{2},0)$.

In the next theorem we determine the decomposition of $L_F(-\frac{5}{2},0)$ into a direct sum of irreducible $L_B(-\frac{5}{2},0)$-modules.

Theorem 30.

$$L_F\left(-\frac{5}{2},0\right)\cong L_B\left(-\frac{5}{2},0\right)\oplus L_B\left(-\frac{5}{2},\bar{\omega}_4\right).$$

Proof. It follows from Theorem 28 that $L_F(-\frac{5}{2},0)$ is an $L_B(-\frac{5}{2},0)$-module, and Proposition 14 implies that it is a direct sum of copies of irreducible $L_B(-\frac{5}{2},0)$-modules $L_B(-\frac{5}{2},0)$ and $L_B(-\frac{5}{2},\bar{\omega}_4)$. Clearly, $\mathbf{1}$ and $e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)}(-1)\mathbf{1}$ are singular vectors for $\hat{\mathfrak{g}}_B$ in $L_F(-\frac{5}{2},0)$ which generate the following $L_B(-\frac{5}{2},0)$-modules:

$$U(\hat{\mathfrak{g}}_B).\mathbf{1}\cong L_B\left(-\frac{5}{2},0\right)\quad\text{and}$$

$$U(\hat{\mathfrak{g}}_B).e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)}(-1)\mathbf{1}\cong L_B\left(-\frac{5}{2},\bar{\omega}_4\right).$$

It follows from relation (3) that the lowest conformal weight of $L_B(-\frac{5}{2},0)$-module $L_B(-\frac{5}{2},0)$ is 0 and of $L_B(-\frac{5}{2},\bar{\omega}_4)$ is 1. Theorem 28 now implies that $\mathbf{1}$ and $e_{\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4)}(-1)\mathbf{1}$ are only singular vectors for $\hat{\mathfrak{g}}_B$ in $L_F(-\frac{5}{2},0)$, which implies the claim of theorem. $\square$

Remark 31. (a) It follows from Theorem 30 that the extension of vertex operator algebra $L_B(-\frac{5}{2},0)$ by $L_B(-\frac{5}{2},\bar{\omega}_4)$, its only irreducible module other then itself, is a vertex operator algebra, which is isomorphic to $L_F(-\frac{5}{2},0)$.

(b) Theorem 30 also implies that $\hat{\mathfrak{g}}_F$-module $L_F(-\frac{5}{2},0)$, considered as a module for Lie subalgebra $\hat{\mathfrak{g}}_B$ of $\hat{\mathfrak{g}}_F$, decomposes into the finite direct sum of $\hat{\mathfrak{g}}_B$-modules. In the case of positive integer levels, such cases are called conformal embeddings of affine Lie algebras in physics and were studied in [5,7,26].

6. Weak $L_F(-\frac{5}{2},0)$-modules from category $\mathcal{O}$

In this section we study the category of weak $L_F(-\frac{5}{2},0)$-modules that are in category $\mathcal{O}$ as $\hat{\mathfrak{g}}_F$-modules. To obtain the classification of irreducible objects in that category, we use methods from [23,2,4] (presented in Section 2.6) and the fact that $L_F(-\frac{5}{2},0)$ contains three copies of $L_B(-\frac{5}{2},0)$ as vertex operator subalgebras. It is proved in [25, Lemma 18] that,

Lemma 32. *Let*

$$p_1(h) = h_{\epsilon_1-\epsilon_2}\left(h_{\epsilon_1+\epsilon_2}+\frac{5}{2}\right),$$
$$p_2(h) = h_{\epsilon_2-\epsilon_3}\left(h_{\epsilon_2+\epsilon_3}+\frac{3}{2}\right),$$
$$p_3(h) = h_{\epsilon_3-\epsilon_4}\left(h_{\epsilon_3+\epsilon_4}+\frac{1}{2}\right),$$
$$p_4(h) = h_{\epsilon_4}(h_{\epsilon_4}-1).$$

Then $p_1, p_2, p_3, p_4 \in \mathcal{P}_0^B$.

Corollary 24 now implies that $p_1, p_2, p_3, p_4 \in \mathcal{P}_0^F$.

Lemma 33. *Let*

$$p_5(h) = h_{\epsilon_3-\epsilon_4}\left(h_{\epsilon_1+\epsilon_2}+\frac{5}{2}\right),$$
$$p_6(h) = h_{\epsilon_2-\epsilon_3}\left(h_{\epsilon_1+\epsilon_4}+\frac{3}{2}\right),$$
$$p_7(h) = h_{\epsilon_3+\epsilon_4}\left(h_{\epsilon_1-\epsilon_2}+\frac{1}{2}\right),$$
$$p_8(h) = h_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)}\left(h_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)}-1\right),$$
$$p_9(h) = h_{\epsilon_3+\epsilon_4}\left(h_{\epsilon_1+\epsilon_2}+\frac{5}{2}\right),$$
$$p_{10}(h) = h_{\epsilon_2-\epsilon_3}\left(h_{\epsilon_1-\epsilon_4}+\frac{3}{2}\right),$$
$$p_{11}(h) = h_{\epsilon_3-\epsilon_4}\left(h_{\epsilon_1-\epsilon_2}+\frac{1}{2}\right),$$
$$p_{12}(h) = h_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)}\left(h_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)}-1\right).$$

Then $p_5,\ldots,p_{12} \in \mathcal{P}_0^F$.

Proof. Using isomorphisms π' and π'', and Lemma 32 we obtain $p_5,\ldots,p_8 \in \mathcal{P}_0^{B'}$ and $p_9,\ldots,p_{12} \in \mathcal{P}_0^{B''}$. Corollary 26 now implies that $p_5,\ldots,p_{12} \in \mathcal{P}_0^F$. □

Proposition 34. *The set*

$$\left\{V_F(0), V_F\left(-\frac{3}{2}\omega_1\right), V_F\left(-\frac{1}{2}\omega_1-\frac{1}{2}\omega_2\right), V_F\left(-\frac{3}{2}\omega_2+\omega_3\right)\right\}$$

provides the complete list of irreducible $A(L_F(-\frac{5}{2},0))$-modules from the category $\mathcal{O}$.

Proof. Corollary 10 implies that highest weights $\mu \in \mathfrak{h}^*$ of irreducible $A(L_F(-\frac{5}{2},0))$-modules from the category $\mathcal{O}$ are in one-to-one correspondence with the solutions of the polynomial equations $p(\mu)=0$ for all $p \in \mathcal{P}_0^F$. Since R^F is the $U(\mathfrak{g}_F)$-submodule of $U(\mathfrak{g}_F)$ generated by the vector u_F under the adjoint action, it is isomorphic to $V_F(2\epsilon_1)$, i.e. $V_F(2\omega_4)$. One can easily check that $\dim R_0^F = 12$, which implies that $\dim \mathcal{P}_0^F = 12$. Since the polynomials $p_1,\ldots,p_{12}$ from Lemmas 32 and 33 are linearly independent elements in $\mathcal{P}_0^F$, they form a basis for $\mathcal{P}_0^F$. The claim of proposition now easily follows by solving the system $p_i(\mu)=0$, for $i=1,\ldots,12$. $\square$

It follows from Zhu's theory that:

Theorem 35. *The set*

$$\left\{L_F\left(-\frac{5}{2},0\right), L_F\left(-\frac{5}{2},-\frac{3}{2}\omega_1\right), L_F\left(-\frac{5}{2},-\frac{1}{2}\omega_1-\frac{1}{2}\omega_2\right), L_F\left(-\frac{5}{2},-\frac{3}{2}\omega_2+\omega_3\right)\right\}$$

provides the complete list of irreducible weak $L_F(-\frac{5}{2},0)$-modules from the category $\mathcal{O}$.

Denote by $\lambda^1=-\frac{5}{2}\Lambda_0$, $\lambda^2=-\frac{5}{2}\Lambda_0-\frac{3}{2}\omega_1$, $\lambda^3=-\frac{5}{2}\Lambda_0-\frac{1}{2}\omega_1-\frac{1}{2}\omega_2$ and $\lambda^4=-\frac{5}{2}\Lambda_0-\frac{3}{2}\omega_2+\omega_3$ the highest weights of irreducible weak $L_F(-\frac{5}{2},0)$-modules from the category $\mathcal{O}$. The following lemma is crucial for proving complete reducibility of weak $L_F(-\frac{5}{2},0)$-modules from the category $\mathcal{O}$.

Proposition 36. *Weights λ^i, $i=1,2,3,4$ are admissible for $\hat{\mathfrak{g}}_F$.*

Proof. It is already proved in Lemma 19 that the weight $\lambda^1=\lambda_1=-\frac{5}{2}\Lambda_0$ is admissible. Similarly, one can easily check that weights λ^2, λ^3 and λ^4 are admissible and that

$$\hat{\Pi}^\vee_{\lambda^2}=\{(\delta-\epsilon_1-\epsilon_3)^\vee,\alpha_2^\vee,\alpha_3^\vee,\alpha_4^\vee,\epsilon_2^\vee\},$$
$$\hat{\Pi}^\vee_{\lambda^3}=\{\alpha_0^\vee,(\epsilon_2-\epsilon_4)^\vee,\alpha_3^\vee,\alpha_4^\vee,\epsilon_3^\vee\},$$
$$\hat{\Pi}^\vee_{\lambda^4}=\{\alpha_0^\vee,\alpha_1^\vee,\epsilon_3^\vee,\alpha_4^\vee,\alpha_3^\vee\}. \quad \square$$

We obtain:

Theorem 37. *Let M be a weak $L_F(-\frac{5}{2},0)$-module from the category $\mathcal{O}$. Then M is completely reducible.*

Proof. Let $L_F(\lambda)$ be some irreducible subquotient of M. Then $L_F(\lambda)$ is an irreducible weak $L_F(-\frac{5}{2},0)$-module and Theorem 35 implies that $\lambda=\lambda^i$, for some $i\in\{1,2,3,4\}$. It follows from Proposition 36 that λ is admissible. Proposition 5 now implies that M is completely reducible. $\square$

7. Decompositions of irreducible weak $L_F(-\frac{5}{2},0)$-modules from category $\mathcal{O}$ into direct sums of irreducible weak $L_B(-\frac{5}{2},0)$-modules

In this section we determine decompositions of irreducible weak $L_F(-\frac{5}{2},0)$-modules from category $\mathcal{O}$, classified in Theorem 35, into direct sums of irreducible weak $L_B(-\frac{5}{2},0)$-modules, and show that these direct sums are finite. We use the fact that these vertex operator algebras have the same conformal vector.

Using relation (3), we can determine the lowest conformal weights of irreducible weak $L_B(-\frac{5}{2},0)$-modules from category $\mathcal{O}$, listed in Proposition 15:

Lemma 38. (1) *The lowest conformal weight of weak $L_B(-\frac{5}{2},0)$-modules $L_B(-\frac{5}{2},-\frac{5}{2}\bar{\omega}_1)$, $L_B(-\frac{5}{2},-\frac{3}{2}\bar{\omega}_3+\bar{\omega}_4)$, $L_B(-\frac{5}{2},\frac{1}{2}\bar{\omega}_1-\frac{3}{2}\bar{\omega}_2)$, $L_B(-\frac{5}{2},-\frac{1}{2}\bar{\omega}_2-\frac{1}{2}\bar{\omega}_3)$ is $-\frac{5}{4}$.*

(2) *The lowest conformal weight of $L_B(-\frac{5}{2},-\frac{7}{2}\bar{\omega}_1+\bar{\omega}_4)$, $L_B(-\frac{5}{2},-\frac{1}{2}\bar{\omega}_3)$, $L_B(-\frac{5}{2},\frac{3}{2}\bar{\omega}_1-\frac{5}{2}\bar{\omega}_2+\bar{\omega}_4)$ and $L_B(-\frac{5}{2},\frac{1}{2}\bar{\omega}_2-\frac{3}{2}\bar{\omega}_3+\bar{\omega}_4)$ is $-\frac{3}{4}$.*

(3) *The lowest conformal weight of $L_B(-\frac{5}{2},-\frac{3}{2}\bar{\omega}_2)$, $L_B(-\frac{5}{2},-\frac{5}{2}\bar{\omega}_2+\bar{\omega}_4)$, $L_B(-\frac{5}{2},-\frac{1}{2}\bar{\omega}_1-\frac{1}{2}\bar{\omega}_2-\frac{1}{2}\bar{\omega}_3)$, $L_B(-\frac{5}{2},-\frac{3}{2}\bar{\omega}_1+\frac{1}{2}\bar{\omega}_2-\frac{3}{2}\bar{\omega}_3+\bar{\omega}_4)$, $L_B(-\frac{5}{2},-\frac{3}{2}\bar{\omega}_1-\frac{1}{2}\bar{\omega}_3)$ and $L_B(-\frac{5}{2},-\frac{1}{2}\bar{\omega}_1-\frac{3}{2}\bar{\omega}_3+\bar{\omega}_4)$ is $-\frac{3}{2}$.*

(4) *The lowest conformal weight of $L_B(-\frac{5}{2},0)$ is 0 and of $L_B(-\frac{5}{2},\bar{\omega}_4)$ is 1.*

Denote by $v^{(2)}$ the highest weight vector of $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$, by $v^{(3)}$ the highest weight vector of $L_F(-\frac{5}{2}, -\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2)$ and by $v^{(4)}$ the highest weight vector of $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_2 + \omega_3)$. Then

Lemma 39. (1) *Relation*

$$f_{\frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)}(0)v^{(2)} = 0$$

holds in $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$.

(2) *Relation*

$$f_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)}(0)v^{(3)} = 0$$

holds in $L_F(-\frac{5}{2}, -\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2)$.

(3) *Relation*

$$f_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)}(0)v^{(4)} = 0$$

holds in $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_2 + \omega_3)$.

Proof. We shall prove claim (1). Claims (2) and (3) can be proved similarly. It is proved in Proposition 36 that the weight $\lambda^2 = -\frac{5}{2}\Lambda_0 - \frac{3}{2}\omega_1$ is admissible and that

$$\hat{\Pi}^{\vee}_{\lambda^2} = \{(\delta - \epsilon_1 - \epsilon_3)^{\vee}, \alpha_2^{\vee}, \alpha_3^{\vee}, \alpha_4^{\vee}, \epsilon_2^{\vee}\}.$$

Proposition 4 now implies that the maximal submodule of the Verma module $M_F(\lambda^2)$ is generated by five singular vectors with weights

$$r_{\delta-\epsilon_1-\epsilon_3}.\lambda^2, \qquad r_{\alpha_2}.\lambda^2, \qquad r_{\alpha_3}.\lambda^2, \qquad r_{\alpha_4}.\lambda^2, \qquad r_{\epsilon_2}.\lambda^2.$$

One can easily check that $f_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)}(0)v^{(2)}$ is a singular vector with weight $r_{\alpha_4}.\lambda^2$, that $f_{\epsilon_4}(0)v^{(2)}$ is a singular vector with weight $r_{\alpha_3}.\lambda^2$, and that $f_{\epsilon_3-\epsilon_4}(0)v^{(2)}$ is a singular vector with weight $r_{\alpha_2}.\lambda^2$. This implies that

$$f_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3-\epsilon_4)}(0)v^{(2)} = 0,$$
$$f_{\epsilon_4}(0)v^{(2)} = 0 \quad \text{and}$$
$$f_{\epsilon_3-\epsilon_4}(0)v^{(2)} = 0$$

holds in $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$. The claim of lemma now follows immediately. □

Theorem 40.

$$(1) L_F\left(-\frac{5}{2}, -\frac{3}{2}\omega_1\right) \cong L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_2\right) \oplus L_B\left(-\frac{5}{2}, -\frac{5}{2}\bar{\omega}_2 + \bar{\omega}_4\right)$$

$$(2) L_F\left(-\frac{5}{2}, -\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2\right) \cong L_B\left(-\frac{5}{2}, -\frac{1}{2}\bar{\omega}_1 - \frac{1}{2}\bar{\omega}_2 - \frac{1}{2}\bar{\omega}_3\right) \oplus L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_1 + \frac{1}{2}\bar{\omega}_2 - \frac{3}{2}\bar{\omega}_3 + \bar{\omega}_4\right)$$

$$(3) L_F\left(-\frac{5}{2}, -\frac{3}{2}\omega_2 + \omega_3\right) \cong L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_1 - \frac{1}{2}\bar{\omega}_3\right) \oplus L_B\left(-\frac{5}{2}, -\frac{1}{2}\bar{\omega}_1 - \frac{3}{2}\bar{\omega}_3 + \bar{\omega}_4\right).$$

Proof. It follows from relation (3) that the lowest conformal weight of $L_F(-\frac{5}{2}, 0)$-modules $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$, $L_F(-\frac{5}{2}, -\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2)$ and $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_2 + \omega_3)$ is $-\frac{3}{2}$. Let us prove claim (1). Let v_λ be any singular vector for $\hat{\mathfrak{g}}_B$ in $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$, with weight $\lambda \in \hat{\mathfrak{h}}^*$. $L_B(\lambda)$ is then a subquotient of weak $L_B(-\frac{5}{2}, 0)$-module $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$, which implies that $L_B(\lambda)$ is an irreducible weak $L_B(-\frac{5}{2}, 0)$-module. Thus, $L_B(\lambda)$ is isomorphic to some weak module listed in Proposition 15. Furthermore, Theorem 28 and Lemma 38 imply that $L_B(\lambda)$ is isomorphic to some weak module from Lemma 38 (3), and that v_λ is in the lowest conformal weight subspace of $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$. Using Lemma 39 (1), one can easily check that $v^{(2)}$ and $f_{\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)}(0)v^{(2)}$ are singular vectors for $\hat{\mathfrak{g}}_B$ in $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$ with weights $-\frac{5}{2}\Lambda_0 - \frac{3}{2}\bar{\omega}_2$ and $-\frac{5}{2}\Lambda_0 - \frac{5}{2}\bar{\omega}_2 + \bar{\omega}_4$, respectively. Moreover, one can easily check that there are no singular vectors for $\hat{\mathfrak{g}}_B$ in $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$ with weights $\bar{\lambda}_1 = -\frac{5}{2}\Lambda_0 - \frac{1}{2}\bar{\omega}_1 - \frac{1}{2}\bar{\omega}_2 - \frac{1}{2}\bar{\omega}_3$, $\bar{\lambda}_2 = -\frac{5}{2}\Lambda_0 - \frac{3}{2}\bar{\omega}_1 + \frac{1}{2}\bar{\omega}_2 - \frac{3}{2}\bar{\omega}_3 + \bar{\omega}_4$, $\bar{\lambda}_3 = -\frac{5}{2}\Lambda_0 - \frac{3}{2}\bar{\omega}_1 - \frac{1}{2}\bar{\omega}_3$ and $\bar{\lambda}_4 = -\frac{5}{2}\Lambda_0 - \frac{1}{2}\bar{\omega}_1 - \frac{3}{2}\bar{\omega}_3 + \bar{\omega}_4$, since $-\frac{5}{2}\Lambda_0 - \frac{3}{2}\omega_1 - \bar{\lambda}_i$ is not a sum of positive roots for $\mathfrak{g}_F$, for $i = 1, 2, 3, 4$.

Thus, $v^{(2)}$ and $f_{\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)}(0)v^{(2)}$ are only singular vectors for $\hat{\mathfrak{g}}_B$ in $L_F(-\frac{5}{2}, -\frac{3}{2}\omega_1)$ which implies that

$$U(\hat{\mathfrak{g}}_B).v^{(2)} \cong L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_2\right) \quad \text{and}$$

$$U(\hat{\mathfrak{g}}_B).f_{\frac{1}{2}(\epsilon_1+\epsilon_2-\epsilon_3-\epsilon_4)}(0)v^{(2)} \cong L_B\left(-\frac{5}{2}, -\frac{5}{2}\bar{\omega}_2 + \bar{\omega}_4\right),$$

and that

$$L_F\left(-\frac{5}{2}, -\frac{3}{2}\omega_1\right) \cong L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_2\right) \oplus L_B\left(-\frac{5}{2}, -\frac{5}{2}\bar{\omega}_2 + \bar{\omega}_4\right).$$

Thus, we have proved claim (1). Using Lemmas 38 and 39, one can similarly obtain that

$$U(\hat{\mathfrak{g}}_B).v^{(3)} \cong L_B\left(-\frac{5}{2}, -\frac{1}{2}\bar{\omega}_1 - \frac{1}{2}\bar{\omega}_2 - \frac{1}{2}\bar{\omega}_3\right) \quad \text{and}$$

$$U(\hat{\mathfrak{g}}_B).f_{\frac{1}{2}(\epsilon_1-\epsilon_2+\epsilon_3-\epsilon_4)}(0)v^{(3)} \cong L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_1 + \frac{1}{2}\bar{\omega}_2 - \frac{3}{2}\bar{\omega}_3 + \bar{\omega}_4\right),$$

and that

$$L_F\left(-\frac{5}{2}, -\frac{1}{2}\omega_1 - \frac{1}{2}\omega_2\right) \cong L_B\left(-\frac{5}{2}, -\frac{1}{2}\bar{\omega}_1 - \frac{1}{2}\bar{\omega}_2 - \frac{1}{2}\bar{\omega}_3\right)$$
$$\oplus L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_1 + \frac{1}{2}\bar{\omega}_2 - \frac{3}{2}\bar{\omega}_3 + \bar{\omega}_4\right)$$

which implies claim (2). Furthermore

$$U(\hat{\mathfrak{g}}_B).v^{(4)} \cong L_B\left(-\frac{5}{2}, -\frac{1}{2}\bar{\omega}_1 - \frac{3}{2}\bar{\omega}_3 + \bar{\omega}_4\right),$$

$$U(\hat{\mathfrak{g}}_B).f_{\frac{1}{2}(\epsilon_1-\epsilon_2-\epsilon_3+\epsilon_4)}(0)v^{(4)} \cong L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_1 - \frac{1}{2}\bar{\omega}_3\right),$$

and

$$L_F\left(-\frac{5}{2}, -\frac{3}{2}\omega_2 + \omega_3\right) \cong L_B\left(-\frac{5}{2}, -\frac{3}{2}\bar{\omega}_1 - \frac{1}{2}\bar{\omega}_3\right) \oplus L_B\left(-\frac{5}{2}, -\frac{1}{2}\bar{\omega}_1 - \frac{3}{2}\bar{\omega}_3 + \bar{\omega}_4\right)$$

which implies claim (3). □

8. Modules for $L_F(n - \frac{7}{2}, 0)$, $n \in \mathbb{N}$

In this section we study the category of $L_F(n - \frac{7}{2}, 0)$-modules, for $n \in \mathbb{N}$. Using Corollary 24 and a certain polynomial from [25], one can easily obtain the classification of irreducible objects in that category. Moreover, one can show that this category is semisimple. We omit the proofs in this section because of their similarity with the proofs given in [25].

The next lemma follows from [25, Lemma 18] and Corollary 24:

Lemma 41. *Let*

$$q(h) = \sum_{\substack{(k_1,\ldots,k_l)\in\mathbb{Z}_+^l \\ \sum k_i = n}} \frac{1}{k_1! 4^{k_1}} \cdot (h_{\epsilon_1} - 2k_2 - \cdots - 2k_l) \cdot \cdots \cdot (h_{\epsilon_1} - 2n + 1)$$
$$\cdot (h_{\epsilon_1-\epsilon_l} - k_{l-1} - \cdots - k_2) \cdot \cdots \cdot (h_{\epsilon_1-\epsilon_l} - k_{l-1} - \cdots - k_2 - k_l + 1)$$
$$\cdot \cdots \cdot h_{\epsilon_1-\epsilon_2} \cdot \cdots \cdot (h_{\epsilon_1-\epsilon_2} - k_2 + 1).$$

Then $q \in \mathcal{P}_0^F$.

It can easily be checked (as in [25, Lemma 21]) that relation $q(\mu) = 0$ for $\mu \in P_+^F$ implies $(\mu, \epsilon_1) \leq n - \frac{1}{2}$. Using results from Section 2.6 we get

Lemma 42. *Assume that $V_F(\mu)$, $\mu \in P_+^F$ is an $A(L_F(n - \frac{7}{2}, 0))$-module. Then $(\mu, \epsilon_1) \leq n - \frac{1}{2}$.*

Moreover, using weight arguments as in [25, Lemma 22] one can prove the converse:

Lemma 43. *Let $\mu \in P_+^F$, such that $(\mu, \epsilon_1) \leq n - \frac{1}{2}$. Then $V_F(\mu)$ is an $A(L_F(n - \frac{7}{2}, 0))$-module.*

We get:

Proposition 44. *The set*

$$\left\{ V_F(\mu) \mid \mu \in P_+^F, (\mu, \epsilon_1) \leq n - \frac{1}{2} \right\}$$

provides the complete list of irreducible finite-dimensional $A(L_F(n - \frac{7}{2}, 0))$-modules.

It follows from Zhu's theory that:

Theorem 45. *The set*

$$\left\{ L_F(n - \frac{7}{2}, \mu) \mid \mu \in P_+^F, (\mu, \epsilon_1) \leq n - \frac{1}{2} \right\}$$

provides the complete list of irreducible $L_F(n - \frac{7}{2}, 0)$-modules.

Furthermore, one can easily check (as in Lemma 19) that highest weights λ of these modules are admissible such that

$$\hat{\Pi}_\lambda^\vee = \{(\delta - \epsilon_1)^\vee, \alpha_1^\vee, \alpha_2^\vee, \alpha_3^\vee, \alpha_4^\vee\}.$$

Using Proposition 5 (as in [25, Lemma 26, Theorem 27]), one can easily obtain:

Theorem 46. *Let M be an $L_F(n - \frac{7}{2}, 0)$-module. Then M is completely reducible.*

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